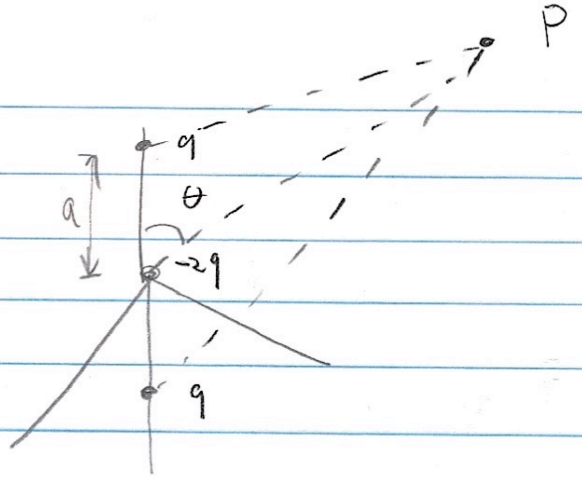
Jackson

3.7 (a)



$$\Phi(r,\theta) = k\left[\frac{q}{|\mathbf{r}_1|} + \frac{q}{|\mathbf{r}_2|} - \frac{2q}{r}\right]$$

where $|\mathbf{r}_1| = [r^2 + a^2 - 2ra\cos\theta]^{1/2}$

$$|\mathbf{r}_2| = [r^2 + a^2 + 2ra\cos\theta]^{1/2} .$$

or $\Phi(r,\theta) = kq\left[\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|} - \frac{2}{r}\right]$

In a, Φ is expanded as

$$\Phi(r,\theta) = \Phi\Big|_{a=0} + a\frac{\partial\Phi}{\partial a}\Big|_{a=0} + \frac{a^2}{2}\frac{\partial^2\Phi}{\partial a^2}\Big|_{a=0} + \cdots$$

$$\Phi\Big|_{a=0} = 0\ \frac{2kq}{r}$$

$$\frac{\partial\Phi}{\partial a} = kq\left(-\frac{1}{2}\right)\left[\left(r^2+a^2-2ra\cos\theta\right)^{-3/2}(2a-2r\cos\theta) + (r^2+a^2+2ra\cos\theta)^{-3/2}(2a+2r\cos\theta)\right]$$

evaluation at $a=0$ yields

$$a\frac{\partial \Phi}{\partial a}\Big|_{a=0} = \frac{-kq}{2}\left[r^{-3}(-2r\cos\theta) + r^{-3}(2r\cos\theta)\right] = 0.$$

$$\frac{\partial^2\Phi}{\partial a^2} = \left(\frac{+kq}{2}\right)\Big[(2a-2r\cos\theta)\left(-\tfrac{3}{2}\right)(r^2+a^2-2ra\cos\theta)^{-5/2}(2a-2r\cos\theta)$$
$$+ (2a+2r\cos\theta)\left(-\tfrac{3}{2}\right)(r^2+a^2-2ra\cos\theta)^{-5/2}(2a+2r\cos\theta)$$
$$+ \cdots \Big]$$

we have neglected terms that vanish at $a=0$

Evaluation at $a=0$ yields

$$\frac{\partial^2\Phi}{\partial a^2}\Big|_{a=0} = \left(\frac{-kq}{2}\right)\left[(2r\cos\theta)^2\left(-\tfrac{3}{2}\right)r^{-5} + (2r\cos\theta)^2\left(-\tfrac{3}{2}\right)r^{-5}\right]$$

$$= \left(-\frac{kq}{2}\right)\left(-\frac{3}{2}\right)(2)(2r\cos\theta)^2 r^{-5}$$

$$= (kq)\left(\frac{3}{2}\right)4r^2\cos^2\theta\, r^{-5}$$

$$= 6kq\cos^2\theta\, r^{-3}.$$

$$\Rightarrow \Phi(r,\theta) = \frac{-2kq}{r} + \frac{a^2}{2}\,6kq\cos^2\theta\, r^{-3} + \cdots$$

$$= kq\left[3\frac{a^2}{r^3}\cos^2\theta - \frac{2}{r}\right] + \cdots$$

$$= -\frac{2kq}{r} + \underline{\frac{3kQ\cos^2\theta}{r^3}} + \cdots$$

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